Deconstructing quantum pulses

1 Introduction

In school, undergraduate courses and public media, light waves are commonly pictured as sine functions that extend indefinitely in time and space. However, a distinct and arguably more intuitive approach to illustrating light is through optical pulses, confined to a specific temporal window. This text discusses the distinctions between these depictions and the methods of describing them. Furthermore, we will see how to extend this description to the quantum level by considering single photon states. Lastly, we explore a specific type of quantum entanglement of two photons within such pulse forms.

2 Modes of the electromagnetic field

We begin by introducing the essential equation that defines the possible shapes of light waves. This equation is known as the wave equation:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} - \nabla^2\mathbf{E} = 0. \tag{1}$$

Any solution $\mathbf{E}(\mathbf{x}, t)$ [1] of this equation is a possible shape of a light wave, which is then often also called 'mode' of the electromagnetic field. The sine-wave picture of light is related to the so called 'continuous waves', which are one set of solutions to Eq. (1), and have the shape $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 \exp(i(\mathbf{x}\cdot\mathbf{k}_0 - \omega_0 t))$ [2]. These waves are characterized by a frequency ω_0 of the wave, a propagation direction $\mathbf{k}_0/|\mathbf{k}_0|$ and a polarization $\mathbf{E}_0/|\mathbf{E}_0|$.

We restrict our following discussion to one spatial point ($\mathbf{x} = \mathbf{0}$) and ignore the polarization ($\mathbf{E} \to E$), since we are only interested in the spectral and temporal profiles of the light-waves. For example the continuous waves then have the form $E(t) = E_0 \exp(i\omega_0 t)$. Note, that such a continuous wave corresponds to exactly one frequency ω_0 (compare Fig. 1).

In contrast it requires multiple frequencies to describe one of the pulsed shapes of light. This is because the temporal shape of the light ($E(t)$) is connected to the frequencies via the Fourier-transformations (FT)

$$E(\omega) = \frac{1}{\sqrt{2}}\int \mathrm{d}\omega E(t)e^{-i\omega t}. \tag{2}$$

This is depicted for the example of a Gaussian shaped pulse in Fig. 1, where then the Fourier-Transformation is also Gaussian. One can interpreter such a FT as a de-

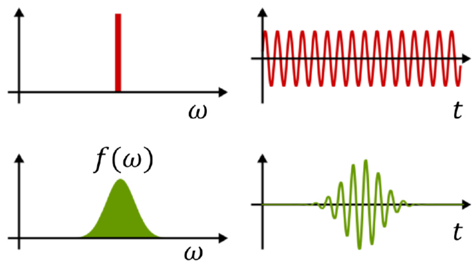

Figure 1: (left): Spectral domain representation, which is given as the Fourier-transformation of the temporal domain representation. (right): Temporal domain representation of a plane wave and a Gaussian pulse.

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composition of $E(t)$ into the continuous waves (compare Fig. 2a). This is possible, because continuous waves form a basis of all solutions to Eq. (1). This becomes also apparent from the inverse Fourier transformation

$$E(t) = \frac{1}{\sqrt{2}}\int \mathrm{d}\omega \underbrace{E(\omega)}_{\text{'weights'}} \underbrace{e^{i\omega t}}_{\text{'CW'}}, \tag{3}$$

which 'constructs' the original pulse from the continuous waves.

However, these are not the only possible basis of the solutions to Eq. (1) and for example the Hermite-Gauss functions

$$\mathrm{HG}_n(t) = \frac{1}{\sqrt{n!\sqrt{\pi}2^n\sigma}}\cdot \mathrm{H}_n\left(\frac{t}{\sigma}\right)\cdot e^{-\frac{t^2}{2\sigma_2}} \tag{4}$$

can also be used to decompose such pulses in an analogous way. In this case the decomposition takes the form:

$$E(t) = \sum_n c_n \mathrm{HG}_n(t) \tag{5}$$

[1] Bold written letters (like $\mathbf{E}$) indicate a vector.

[2] Note, that we use complex number to describe the field. The field is then given by $Re(E(t))$. Because of Euler's formula ($e^{i\phi} = \cos\phi + i\sin\phi$) the continuous waves are then cos-functions.

with in general complex coefficients c_k (compare Fig. 2b). Note, that in this case the decomposition runs over a discrete set of coefficients, whereas the Fourier transformation runs over the continues frequency.

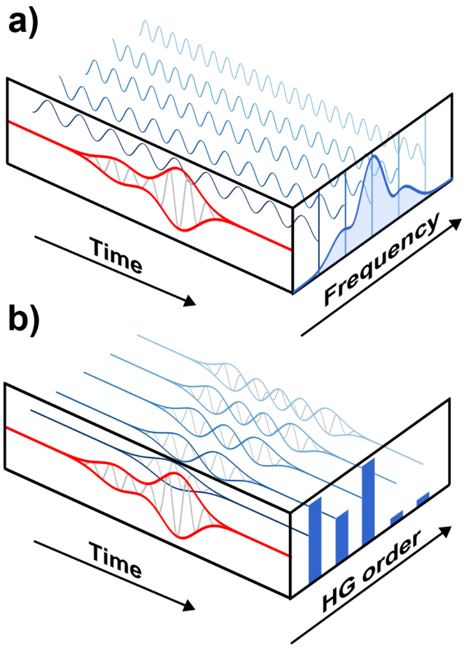

Figure 2: a) Schematic depiction of a Fourier-transformation. The temporal shape is decomposed in the basis of continuous waves. b) Decomposition of the pulse into the basis of Hermit-Gaussian shaped pulses.

3 One photon pulse

The plane waves and pulses discussed so far are still following a fully classical description of light. However, fundamentally light consist of photons and has quantum mechanical properties. To describe this 'quantization' of light into a discrete number of particles, each 'mode' (solution of Eq. (1)) is treated as a quantum harmonic oscillator. This means that for each mode a creation operator $\hat{a}^\dagger$ and an annihilation operator $\hat{a}$ are defined by:

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{6}$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,. \tag{7}$$

In this $|n\rangle$ describes a state which has n photons in the described mode. Therefore, creation and annihilation operators are 'adding' and 'subtracting' exactly one photon from said mode. The complete light field, which consists of infinitely many of such modes then consist of infinitely many quantum harmonic oscillators. Since the continuous waves are a complete basis, every state of light can be represented by the operators $\hat{a}^\dagger(\omega)$, which describe the creation of a photon with frequency ω.

For example, to describe a single photon in a pulsed mode (with a spectrum $E(\omega)$) we can use these operators and construc the state

$$|1\rangle_{E(\omega)} = \underbrace{\int \mathrm{d}\omega E(\omega)\hat{a}^\dagger(\omega)}_{\hat{E}^\dagger :=}|0\rangle\,. \tag{8}$$

The resulting state corresponds to a superposition of single-photon states with different frequencies $\hat{a}^\dagger(\omega)|0\rangle$, weighted by the spectrum. Note however that we also can define the operator $\hat{E}^\dagger$ which then directly describes the creation of a photon in the pulsed mode[3].

4 Spectrally entangled photons

The quantum nature of light only really reveals itself once multiple photons are considered together, since then effects like entanglement can occur. Here, we will consider a special type of entanglement between two photons. Namely we are speaking about frequency entanglement, which reveals itself in correlation in the frequencies of pulses. Such states can for example be generated in so called 'non-linear' crystals, in which an input photon (called pump) can split into two photons (called signal and idler). A general pure two-photon state created in such a process can be written as

$$|\Psi\rangle = \int \mathrm{d}\omega_s \mathrm{d}\omega_i \,\mathrm{JSA}(\omega_s,\omega_i)\,\hat{a}^\dagger(\omega_s)\hat{b}^\dagger(\omega_i)\;|0,0\rangle \tag{9}$$

Here, we use to different operators $\hat{a}$ and $\hat{b}$ to account for the fact that signal and idler photons have different polarization. Note, that this formulation is very similar to Eq. (8), but here the 2D function $\mathrm{JSA}(\omega_s,\omega_i)$ is determining the weights in the superposition of the two-photon states. This function is called 'joint spectral amplitude' and it describes the correlations between the frequencies of the signal and idler photons. A common way to reveal these correlations is by performing a so called Schmidt decomposition, which is similar to the 1D decomposition of $E(t)$ we have discussed previously, however here it is decomposing a 2D function. This Schmidt decomposition of the JSA gives

$$\mathrm{JSA}(\omega_s,\omega_i) = \sum_k r_k A_k(\omega_s)B_k(\omega_i), \tag{10}$$

where r_k are so called Schmidt coefficients and represent the 'weights'. The sets of functions $\{A_k(\omega_s)\}$ and $\{B_k(\omega_i)\}$ form complete bases of the signal and idler spaces. Crucially, in a Schmidt decomposition these functions are paired (e.g. $A_1 \longleftrightarrow B_1$). This is illustrated for the example of 2D Gaussian shaped JSA in Fig. 3.

[3]Mathematically this corresponds to a basis change. For example we can also work in the basis of Hermite-Gaussian modes.

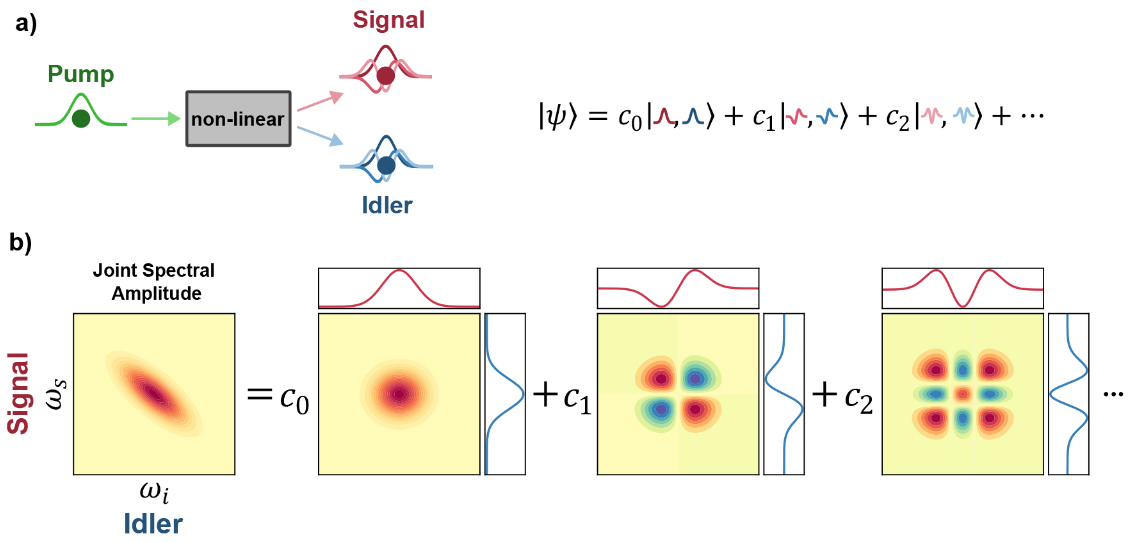

Figure 3: a) Schematic of the generation of a photon pair in a non-linear crytal. b) Depiction of the Schmidt decomposition of a joint spectral amplitude

In this case the Schmidt decomposition results in pairs of Hermite-Gaussian function.

Inserting the decomposed JSA into Eq. (9) then allows to rewrite the state in terms of operators corresponding to the pulsed mode:

$$|\Psi\rangle = \sum_k r_k \hat{A}_k^\dagger \hat{B}_k^\dagger |0,0\rangle , \qquad (11)$$

where $\hat{A}_k^\dagger = \int \mathrm{d}\omega A(\omega)_k \hat{a}^\dagger(\omega)$ and $\hat{B}_k^\dagger = \int \mathrm{d}\omega B(\omega)_k \hat{b}^\dagger(\omega)$ are the generation operators for the pulsed modes.

From Eq. (11) the entanglement becomes apparent, because it basically says: "When the idler is detected in, for example, mode B_1 then the signal photon collapses to mode A_1". Only, for the case when only one Schmidt coefficient is non-zero the state is not entangled, since in this case the detection of the idler does not influence the signal.