LEARN: HEISENBERG PRINCIPLE - Advanced

One of the most famous principles of QP is the Heisenberg uncertainty principle, firstly introduced in 1927 by the German physicist Werner Heisenberg. The principle sets a fundamental limit to the accuracy with which the values of certain pairs of physical quantities, named complementary or canonically conjugate variables, e.g. the position and the momentum of a particle, can be known. In physics *knowing* means *measuring*, and the Heisenberg principle is indeed related to the measurement, although the connection between the two is subtle. In fact, for a long time it was mistakenly identified with another physical effect, known as the *observer effect.* The latter refers to the situation in which measuring a physical quantity inevitably disturbs the system. A typical example is the measurement of the pressure of a car tyre: it's very difficult to perform it without letting some air escape, that is, without altering the very same property that we want to measure, the pressure itself. Such effect was initially used by Heisenberg himself to explain the uncertainty in the Quantum World. Today we know that the uncertainty principle means something different: it's related to intrinsic features of all quantum systems, and neither experimental accuracy nor the technology exploited to perform the measurement have anything to do with it. Therefore, even if undoubtedly QP shed new light to the concept of ideal measurement, typical of Classical Physics, we must avoid confusing the observer effect with the Heisenberg uncertainty principle. While the latter is a purely quantum phenomenon, the former exists also for classical systems.

The mathematical description of measurement of an observable M encompasses a Hermitian operator $\hat{M}$, a set of possible results $\{m_\ell\}$ with the associated projectors $\{|\ell\rangle\}_{\mathscr{H}}$, and the probability $p^{M}_{|\Psi\rangle}(m_\ell)$ of obtaining the measurement outcome ℓ, given the state $|\Psi\rangle$ (see the Quest entry measurement). These, in turn, allow us to define its mean value $\overline{M} = \langle\Psi|\hat{M}|\Psi\rangle$. Let us consider an observable A and the associated operator $\hat{A}$, with mean value $\overline{A} = \langle\Psi|\hat{A}|\Psi\rangle$. We define the standard deviation as

$$\Delta A = \sqrt{\langle\Psi|(\hat{A}-\overline{A})^2|\Psi\rangle} = \sqrt{\langle(\hat{A}-\overline{A})^2\rangle}\,.$$

This is a positive real number quantifying the statistical fluctuations of the measurement outcomes. Therefore, the smaller ΔA, the higher the accuracy of the measurement. Given two observables, A and B, it's possible to show that

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|\,, \qquad (1)$$

where $[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator between $\hat{A}$ and $\hat{B}$, usually different from zero due to the structure of the Hilbert space where the operators act[1].

Inequality (1) represents the Heisenberg principle in its most general form: the more precisely determined A is, the more undetermined B is, since the smaller ΔA, the bigger ΔB, and vice versa. The commutator between two operators can be *trivial,* that is, equal to zero, $[\hat{A},\hat{B}] = 0$, or equal to a third operator, $[\hat{A},\hat{B}] = \hat{C}$. A common example is a commutator equal to the identity times a scalar, as is the case for the position operator, $\hat{X}$, and the corresponding momentum operator, $\hat{P}_x$, for which $[\hat{X},\hat{P}_x] = i\hbar\hat{\mathbb{1}}$. This is known as the canonical commutation relation. From equation (1) it follows that

$$\Delta X \Delta P \geq \tfrac{1}{2}\left|\langle[\hat{X},\hat{P}]\rangle\right| = \tfrac{\hbar}{2}; \qquad (2)$$

How well-determined the position X is limits how well-determined the momentum P_x can be, and vice versa.

[1] For instance, if we think about the matrix representation of the operators, it is easy to see that the product of two arbitrary matrices often depends on the order in which it is performed, so that their commutator is different from zero.

The uncertainty principle is a consequence of the non-commutativity between operators, since ΔA and ΔB can be both equal to zero only if $[\hat{A}, \hat{B}] = 0$. When this is the case, $\hat{A}$ and $\hat{B}$ are named ***commuting operators***, and it can be shown that they have the same eigenstates. In other words, if $\hat{A}$ and $\hat{B}$ commute, there exists a basis $\{|\gamma\rangle\}_{\mathscr{H}}$ that diagonalises them simultaneously, i.e.,

$$\hat{A} = \sum_{\gamma} \alpha_{\gamma} |\gamma\rangle\langle\gamma| \text{ and } \hat{B} = \sum_{\gamma} \beta_{\gamma} |\gamma\rangle\langle\gamma|.$$

Therefore, relation (2) implies that no quantum state can be an eigenstate of the position and of the momentum simultaneously.

As we can learn in the quantum measurement entry of Quest, when the observed system is in an eigenstate of an observable, measuring that observable will always yield a deterministic outcome. This means that, in that case, all the measurements of observables associated to commuting operators will be deterministic as well, and hence we can state that both quantities are determined. Instead, when the system is in an eigenstate of some observable, but we measure another observable associated to an operator which does not commute with the former, we'll find a probabilistic answer.